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On locality, correlation and hidden variables

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Abstract. Locality conditions for hidden-variables descriptions of the Bohm–EPR thought experiment are shown to be unnecessary as the underlying structure is already local. An objective, local hidden-variables theory is given.

1. Introduction

Ever since the genesis of quantum mechanics there has been much debate over the metaphysics required by such a theory and over the necessity of the form of this empirically successful theory (as much as one exists). Specifically, three aspects of the metaphysics implicit and explicit in the pre-quantum physics were brought into question: objectivity—the existence of matter and its physical properties independent of observation, locality—the requirement that there be no action at a distance, and determinism—the notion that, given the state of the universe at any time, all subsequent states can be predicted with certainty. Locality requires some explanation, partly because of the poor nomenclature. What is required in a local theory is that space–time be divisible into regions between which no signal travelling at or below the speed of light may pass. Such regions are said to be separated in a space-like manner. The consequence of this requirement that is of interest here is that nothing that happens in a region of space–time can influence events in those regions from which that region is space-like separated. The formulation of the non-relativistic quantum theory obviously violates the principle of locality, with the concept of wave-packet collapse in measurement, and also casts doubt upon objectivity and determinism because it only describes the outcomes of measurements and then only in a statistical way. This led to the positivistic 'Copenhagen interpretation' whose leading proponent Bohr, claimed that 'the quantum of action entails . . . the necessity of a final renunciation of the classical ideal of causality and a radical revision of our attitude towards the problem of physical reality' (Bohr 1935, p 697).

Einstein, Podolsky, and Rosen (EPR) (1935), however, argued that there was no clear need to change our metaphysics but rather that it was the quantum theory itself which was causing the difficulties. They did this by requiring that a theory of physics should provide a description of all the physical properties of a system. Quantum mechanics clearly does not satisfy this condition if we accept that all theoretically (though not necessarily simultaneously) measurable properties have an objective existence. This led EPR to the conclusion that the quantum description of nature is incomplete. Thus it appeared that by supplementing the quantum theory with extra, so-called 'hidden-variables', a theory which was compatible with at least some of the above principles could be formulated. Objectivity and locality were given priority in this quest as many regarded determinism as unnecessary and even undesirable.

A thought experiment based upon the work of EPR, produced by Bohm (1951), became central to the development of the discussion of hidden-variables theories. Bohm considered two spin-half particles produced in the singlet state which were then separated without disturbing their spins. Any measurement of the spin of one of these particles will enable us to predict the component of spin in the same direction for the other particle. But we may also measure the component of spin of this second particle in any other direction. Therefore, if we accept objectivity and locality a spin-half particle must have at least two definite components of spin. Quantum mechanics cannot fully describe this.

The next major step came with Bell (1964) showing that no deterministic, objective, local hidden-variables theory for the Bohm–EPR experiment could reproduce all the predictions of quantum mechanics for that experiment. This was followed by a series of papers by Clauser *et al* (1969), Bell (1971), and Clauser and Horne (1974) where the condition of determinism was dropped but the same conclusion was reached. It should be mentioned that only ergodic theories were considered by these authors although this was only done implicitly.

2. Constraints on the hidden variables

The method employed was to consider the hidden variables to be distributed over any set of values but with a fixed distribution function. These hidden variables were considered as being objective properties of the entire particle-measuring device system.

Expectations of the outcomes of experiments were defined as being the integral (or sum for discrete cases) of the products of the conditional expectations and the distribution function, where limits upon the conditional expectations had been derived in line with the other metaphysical conditions placed upon the system. Inequalities involving the expectations were then derived which were shown to be violated in some cases by the quantum expectations. It is the limits placed upon the conditional expectations that are of most interest here.

To see how these were formulated let us examine the structure upon which they were imposed. In doing so, an idealised version of the Bohm-EPR experiment will be employed. We consider only those situations in which both particles enter the analysers and that the analysers accurately record the components of spin which are being measured. Thus any indeterminacy in the conditional expectations must arise from the period of interaction between the two particles, which is assumed to cease before they enter the analysers—this is the locality requirement. Clearly, if a hidden-variables theory could be shown to give the same predictions as quantum mechanics in this case then any generalisation to the non-ideal case would not introduce discrepancies between the two.

Let us denote the hidden variables by λ. These are distributed over the set Λ with frequency $f(\lambda)$. Also, let the outcomes of measurements be denoted by A_a and B_b where the upper case letter refers to the measuring device and the subscript gives the direction of measurement. Then

$$A_a(B_b) = \begin{cases} +1 & \text{spin-up recorded in direction } \boldsymbol{a}(\boldsymbol{b}) \\ -1 & \text{spin-down recorded in direction } \boldsymbol{a}(\boldsymbol{b}). \end{cases} \tag{1}$$

The expectations are related to the conditional expectations by

$$E[A_a(B_b)] = \int_\Lambda E[A_a(B_b)|\lambda]f(\lambda)\,\mathrm{d}\lambda$$

and

$$E[A_aB_b] = \int_\Lambda E[A_aB_b|\lambda]f(\lambda)\,\mathrm{d}\lambda. \tag{2}$$

These conditional expectations are further related by

$$E[A_aB_b|\lambda] = E[A_a|\lambda]E[B_b|\lambda] + \mathrm{cov}\,[A_a, B_b|\lambda] \tag{3}$$

where $\mathrm{cov}\,[A_a, B_b|\lambda]$ is the conditional covariance.

Since the particles were prepared in the singlet state we have

$$B_a = -A_a = A_{-a} \tag{4}$$

thus (using (1) and substituting into (3))

$$\begin{aligned} E[A_aB_{-a}|\lambda] &= E[A_a|\lambda]E[A_a|\lambda] + V[A_a|\lambda] \\ &= 1 \end{aligned} \tag{5}$$

where, by definition, the conditional variance, $V[A_a|\lambda]$, is equal to $\mathrm{cov}\,[A_a, A_a|\lambda]$. This is the most general structure for an ergodic, objective hidden-variables theory.

For a deterministic theory we require that

$$|E[A_a(B_b)|\lambda]| = 1, \qquad \text{and} \qquad V[A_a(B_b)|\lambda] = 0 \tag{6}$$

which is consistent with (5). Also we have the general result from probability theory that

$$|\mathrm{cov}\,[A_a, B_b|\lambda]| \leq \{V[A_a|\lambda]V[B_b|\lambda]\}^{1/2} \tag{7}$$

giving us (from (3))

$$E[A_aB_b|\lambda] = E[A_a|\lambda]E[B_b|\lambda] \tag{8}$$

Suppes and Zanotti (1976) first proved that (8) is a sufficient condition for determinism, however (using (5)) we can easily see that this is so.

Equation (8) is, oddly enough, precisely what Bell (1964, 1971) and Clauser *et al* (1969) required of a hidden-variables theory for it to be local, not deterministic. Recently Selleri and Tarozzi (1980), and Garuccio and Rapisarda (1981) have shown that (8) is in fact not required of a local probabilistic theory but this only poses the question of what condition is necessary. The answer is easily provided by an example from the macroscopic world. Suppose we take a sample of children and measure their ages and their heights. There is no question of one of these measurements influencing the outcome of the other measurement for any individual. This requirement of independent measurements is exactly that which we desire of a local theory. However, in analysing the data thus obtained for any subset of this group (say redheads) the equations used would be exactly those given in (2) and (3) without any extra requirements to meet locality. Thus these equations are inherently local and locality conditions are 'red herrings'. The only equations needed to be satisfied by an objective, local hidden-variables theory of the Bohm–EPR experiment are (1)–(4) and (7).

3. A hidden-variables theory

Is it then possible to construct such a theory? The answer is yes and one will be given. The necessity and uniqueness of the theory will not be considered here as our aim is merely to show what is possible.

Let us consider that spin-half particles do have a definite direction of spin, $\boldsymbol{\sigma}$, where $\boldsymbol{\sigma}$ is a unit vector. Further consider that a measurement of the spin disturbs it so that the spin after the measurement points in the direction for which it gave a positive value. That is

$$\boldsymbol{\sigma} \rightarrow A_a \boldsymbol{a} \tag{9}$$

where $\boldsymbol{a}$ is the unit vector in the direction of measurement.

Now we have to find some form for the conditional expectations. Quantum mechanics predicts that the expectation for a particle measured in the direction $\boldsymbol{b}$, given that the spin has been measured as up in the direction $\boldsymbol{a}$, is

$$E[A_b|A_a = 1] = \boldsymbol{a} \cdot \boldsymbol{b}. \tag{10}$$

Given (9) and (10) we propose that the conditional expectation for a single particle be given by

$$E[A_a|\boldsymbol{\sigma}] = \boldsymbol{\sigma} \cdot \boldsymbol{a}. \tag{11}$$

Using (5) and given that all the vectors are of unit length, the conditional variance becomes

$$V[A_a|\boldsymbol{\sigma}] = 1 - (\boldsymbol{\sigma} \cdot \boldsymbol{a})^2 = |\boldsymbol{\sigma} \times \boldsymbol{a}|^2. \tag{12}$$

We can always choose our coordinates such that

$$\boldsymbol{a} = \boldsymbol{k} \qquad \boldsymbol{\sigma} = \sin\phi \cos\psi \boldsymbol{i} + \sin\phi \sin\psi \boldsymbol{j} + \cos\phi \boldsymbol{k}. \tag{13}$$

If we require $\boldsymbol{\sigma}$ to be distributed uniformly over all possible values, the expectation is given by

$$\begin{aligned} E[A_a] &= \frac{1}{4\pi} \int_0^{\pi} \mathrm{d}\phi \int_{-\pi}^{\pi} \mathrm{d}\psi \sin\phi \cos\phi \\ &= 0 \end{aligned} \tag{14}$$

Thus we are in agreement with quantum mechanics in the one-particle case.

For the Bohm–EPR experiment, we require that the spins of the two particles be anti-correlated, that is

$$\boldsymbol{\sigma}_A = -\boldsymbol{\sigma}_B \equiv \boldsymbol{\sigma}. \tag{15}$$

A successful hidden variables theory must agree with the quantum expectation which is

$$E[A_a B_b] = -\boldsymbol{a} \cdot \boldsymbol{b} = -\cos\theta \tag{16}$$

where θ is the small angle between $\boldsymbol{a}$ and $\boldsymbol{b}$. Thus (using (2) and (3))

$$\begin{aligned} &\frac{1}{4\pi} \int_0^{\pi} \mathrm{d}\phi \int_{-\pi}^{\pi} \mathrm{d}\psi \operatorname{cov}[A_a, B_b|\boldsymbol{\sigma}] \sin\phi \\ &\qquad = -\cos\theta - \frac{1}{4\pi} \int_0^{\pi} \mathrm{d}\phi \int_{-\pi}^{\pi} \mathrm{d}\psi \sin\phi (\boldsymbol{\sigma} \cdot \boldsymbol{a})(-\boldsymbol{\sigma} \cdot \boldsymbol{b}) \\ &\qquad = -\tfrac{2}{3} \cos\theta \end{aligned} \tag{17}$$

where the coordinates have been chosen to satisfy (13) and give

$$\boldsymbol{b} = \sin\theta \boldsymbol{i} + \cos\theta \boldsymbol{k}. \tag{18}$$

To find the conditional covariance we write it in the following way

$$\operatorname{cov}[A_a, B_b|\boldsymbol{\sigma}] = \rho(A_a, B_b, \boldsymbol{\sigma})|\boldsymbol{\sigma}\times\boldsymbol{a}|\,|\boldsymbol{\sigma}\times\boldsymbol{b}| \tag{19}$$

where ρ is the correlation coefficient. The simplest correlation coefficient is one which depends on θ only. Using (13), (17), (18) and (19) this gives

$$\rho(\theta) = -2\cos\theta/3F(\theta) \tag{20}$$

where

$$F(\theta) = \frac{1}{4\pi}\int_0^\pi d\phi \int_{-\pi}^{\pi} d\psi\,[1-(\sin\theta\sin\phi\cos\psi + \cos\theta\cos\phi)^2]^{1/2}\sin^2\phi.$$

All that remains to be shown is (from (7) and (19)) that $|\rho(\theta)| \leqslant 1$. Unfortunately the integral in (20) is elliptic but it has been solved numerically and the results are shown in figure 1. This final condition is clearly satisfied.

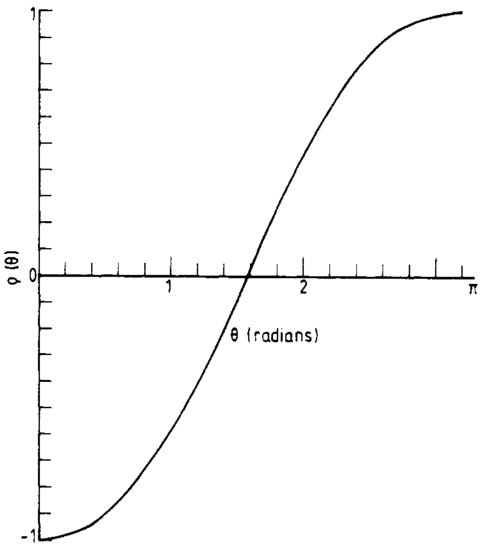

Figure 1. Graph showing results of numerical solution of the integral in equation (20).

The conditional probabilities for the one- and two-particle cases are given by

$$P[A_a = \alpha|\boldsymbol{\sigma}] = \tfrac{1}{2}(1 + \alpha\boldsymbol{\sigma}\cdot\boldsymbol{a}), \qquad P[B_b = \beta|\boldsymbol{\sigma}] = \tfrac{1}{2}(1 - \beta\boldsymbol{\sigma}\cdot\boldsymbol{b}), \tag{21}$$

and

$$P[A_a = \alpha \cap B_b = \beta|\boldsymbol{\sigma}] = P[A_a = \alpha|\boldsymbol{\sigma}]P[B_b = \beta|\boldsymbol{\sigma}] + \tfrac{1}{4}\alpha\beta\operatorname{cov}[A_a, B_b|\boldsymbol{\sigma}],$$

where $\alpha, \beta \in \{-1, 1\}$. It should be noted that given $\boldsymbol{\sigma}$ and its (unquestionably feasible) distribution and equations (14) and (16) a theorem by Kolmogorov (1933) ensures that the conditional expectations and probabilities exist and are unique. Our choice of the dependence of the correlation coefficient is therefore vindicated.

This hidden-variables theory has several notable features. Most importantly it embodies two of the ideas which the introduction of quantum mechanics led us to consider. Firstly there is indeterminancy and secondly we have the notion of the measuring device disturbing the system, albeit in a predictable fashion. On the other hand, for the Bohm–EPR type experiments there is no need to reject the pre-existing ideas of reality and locality. Also, by introducing a hidden variable we have not ascribed a new physical property to the system but merely reaffirmed that we cannot simultaneously measure all the properties of the system. That is, we cannot measure the initial direction of the spin but only its post-measurement direction.

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